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Degree of Approximation by Monotone Polynomials II[1]

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In this note we shall study the degree of approximation of a continuous function f, defined on $[-1,+1]$, by polynomials P_n of degree (at most) n, which satisfy the additional restriction

$$P_n^{(k)}(x) \geqslant 0, \qquad -1 \leqslant x \leqslant 1, \tag{1}$$

where k is a given positive integer. Let $E_n(f)$ denote the degree of approximation of f by arbitrary polynomials P_n of degree n, and let $E_n{}^*(f)$ be the degree of approximation of f by polynomials P_n restricted by (1). It is clear that $E_n(f) \leqslant E_n{}^*(f)$. The problem is, how much larger can $E_n{}^*(f)$ be than $E_n(f)$, for a function f, which satisfies $f^{(k)}(x) \geqslant 0$ on $[-1,1]$. Answering a question raised in our previous paper [2], we show that there exist functions f for which

$$\limsup_{n\to\infty} \frac{E_n^*(f)}{E_n(f)} = +\infty \tag{2}$$

A function f with this property will be constructed by a method of condensation of singularities. The proof below seems to be nontrivial, since the methods of the gliding hump, and of sets of first category, do not apply here (at least, not in an obvious fashion). In fact, they do not, generally, apply to the quotient of two semi-norms.

Our main result is

THEOREM. *There exists, for each $k = 1, 2, \ldots$, a function f with $f^{(k)}(x) \geqslant 0$ on $[-1,+1]$, for which the ratio $E_n{}^*(f)/E_n(f)$ is unbounded for $n \to \infty$.*

We need the following:

LEMMA. *For each k and each $b > 0$, there exist a polynomial g with $g^{(k)}(x) \geqslant 0$ on $[-1,+1]$ and a polynomial P of degree $3k + 4$, with the following properties*

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(*for the supremum norm on* $[-1,+1]$:

$$\|g\| < 1, \qquad \|P\| < 1, \tag{3}$$

$$g^{(k)}(0) = 0, \qquad P^{(k)}(0) < 0, \tag{4}$$

$$|P^{(k)}(0)| > b\|g - P\| > 0. \tag{5}$$

Proof. We establish this first for a continuously differentiable function G, instead of the polynomial g; G will satisfy (3)–(5), and have some additional properties. If k is odd, we put $P(x) = (x^2 - a^2)^{k+2} x^k$, where $a > 0$ is small. Then $P^{(k)}(0) = k!(-1)^k a^{2k+4} < 0$. All derivatives of P are linear combinations, with positive coefficients, of terms of type $(x^2 - a^2)^i x^j$; therefore they are positive on $[a, 1]$. In particular, $P^{(k+1)}(x) \geqslant 0$ on this interval, while $P^{(k)}(x)$, being an even polynomial, is positive outside of $[-a, a]$. Moreover, $\|P\| < 1$.

Let

$$G(x) = \begin{cases} 0 & \text{on } [-a, a], \\ P(x) & \text{elsewhere on } [-1, +1]. \end{cases} \tag{6}$$

It is clear that G is $(k+1)$ times continuously differentiable, with $G^{(k)}(x) \geqslant 0$ on $[-1, +1]$, $G^{(k+1)}(x) \geqslant 0$ on $[0, 1]$. Since

$$\|G - P\| = \max_{|x| \leqslant a} |P(x)| \leqslant a^{3k+4},$$

we have (5) for all sufficiently small a. Similarly, if k is even, we take $P(x) = (x^2 - a^2)^{k+2} x^{k-2}$, and define G in the same way. We note that

$$G \text{ is odd if } k \text{ is odd, and even if } k \text{ is even.} \tag{7}$$

We now approximate G by a polynomial of high degree. Since we shall use Bernstein polynomials, we replace G by the function $H(x) = G(2x - 1)$, defined for $0 \leqslant x \leqslant 1$. Let

$$Q_n(x) = B_n(x) - \frac{1}{k!} B_n^{(k)}(\tfrac{1}{2})\, x^k,$$

where $B_n(x)$ is the Bernstein polynomial of degree n of H. With the notation $p_{n\nu}(x) = \binom{n}{\nu} x^\nu (1 - x)^{n-\nu}$, $\nu = 0, \ldots, n$, $n = 1, 2, \ldots$, we have ([1], p. 14):

$$B_\nu^{(k)}(x) = n \ldots (n - k + 1) \sum_{\nu=0}^{n-k} \left\{ H\left(\frac{\nu + k}{n}\right) - \frac{k}{1} H\left(\frac{\nu + k - 1}{n}\right) + \ldots \right.$$

$$\left. + (-1)^k H\left(\frac{\nu}{n}\right) \right\} p_{n-k,\nu}(x), \tag{8}$$

$$\begin{cases} B^{(k+1)}(x) = n \dots (n-k) \sum_{\nu=0}^{n-k-1} \Delta_{n\nu} p_{n-k-1,\nu}(x), \\ \Delta_{n\nu} = H\left(\frac{\nu+k+1}{n}\right) - \frac{k+1}{n} H\left(\frac{\nu+k}{n}\right) + \dots + (-1)^{k+1} H\left(\frac{\nu}{n}\right). \end{cases} \tag{9}$$

From (8) we see that $B_n^{(k)}(x) \geqslant 0$, $0 \leqslant x \leqslant 1$. Since $H(x) = 0$ for $|x - \frac{1}{2}| < \frac{1}{2}a$, all values of H in (8) vanish for $|(\nu/n) - \frac{1}{2}| < \frac{1}{4}a$, if n is sufficiently large. Using an estimate of the $p_{n\nu}$ ([1], p. 15), we have

$$B_n^{(k)}(\tfrac{1}{2}) \leqslant 2^k n^k \sum_{|(\nu/n)-\frac{1}{2}| \geqslant \frac{1}{4}a} p_{n\nu}(x) \leqslant \frac{A}{n}, \tag{10}$$

where A is some constant.

In (9), we combine together terms indexed ν and $\nu_1 = n - k - 1 - \nu$. Because of (7), $H(1-x) = -H(x)$ if k is odd, and $H(1-x) = H(x)$ if k is even. For an odd k,

$$H\left(\frac{\nu_1+k+1}{n}\right) = -H\left(\frac{n-k-1-\nu_1}{n}\right) = -H\left(\frac{\nu}{n}\right) = -(-1)^{k+1} H\left(\frac{\nu}{n}\right),$$

$$H\left(\frac{\nu_1+k}{n}\right) = -(-1)^k H\left(\frac{\nu+1}{n}\right),$$

and so on; hence

$$\Delta_{n\nu} = -\Delta_{n,\nu_1}.$$

This relation also holds if k is even. Thus, $B_n^{(k+1)}(x)$ is equal to

$$\sum_{\nu \geqslant \frac{1}{2}m} \Delta_{n\nu}\{p_{m\nu}(x) - p_{m,m-\nu}(x)\}, \tag{11}$$

where $m = n - k - 1$. But for $\nu \geqslant \frac{1}{2}m$ and $x \leqslant \frac{1}{2}$,

$$p_{m\nu}(x) - p_{m,m-\nu}(x) = \binom{m}{\nu} x^{m-\nu}(1-x)^{m-\nu}\{x^{2\nu} - (1-x)^{2\nu}\} \leqslant 0.$$

The terms of the sum (11) are equal to zero if ν is close to $\frac{1}{2}m$, and $\Delta_{n\nu} \geqslant 0$ for the other ν, since $H^{(k+1)}(x) \geqslant 0$ on $[\frac{1}{2}, 1]$. Hence we obtain that $B_n^{(k+1)}(x) \leqslant 0$ for $x \leqslant \frac{1}{2}$. Thus $B_n^{(k)}$ decreases on $[0, \frac{1}{2}]$, and by symmetry, increases on $[\frac{1}{2}, 1]$. Therefore $B_n^{(k)}(\frac{1}{2})$ is the minimal value of $B_n^{(k)}$ on $[0,1]$. It follows that $Q_n^{(k)}(x) \geqslant 0$; also by (10), $Q_n \to H$ for $n \to \infty$. If we take $g(x) = Q_n(\frac{1}{2}(1+x))$, we shall have $g^{(k)}(x) \geqslant 0$ on $[-1, +1]$, and $\|g - G\|$ will be arbitrarily small, hence (5) will hold.

Proof of Theorem. We construct, according to the lemma, a sequence of pairs f_i, P_i, $i = 1, 2, \dots$, which correspond to increasingly large b_i. Each P_i

is a polynomial of degree $3k+4$, each f_i is an increasing polynomial of degree N_i, and

$$\|f_i\| < 1, \qquad \|P_i\| < 1, \tag{12}$$

$$f_i'(0) = 0, \qquad P_i'(0) < 0, \tag{13}$$

$$|P_i^{(k)}(0)| > b_i\|f_i - P_i\| > 0. \tag{14}$$

We can assume that $3k+4 \leqslant N_1 < \ldots < N_n < \ldots$, and take $b_i = (2i+2)N_{i-1}^{2k}$. The function f of the theorem will be given by the series

$$f = \sum_{i=1}^{\infty} c_i f_i, \tag{15}$$

where the $c_i > 0$ satisfy

$$c_i \leqslant i^{-2} M_i, \qquad M_i = \max(\|f_i\|, \ldots, \|f_i^{(k)}\|), \qquad i = 1, 2, \ldots, \tag{16}$$

and

$$\sum_{i=n+1}^{\infty} c_i \leqslant c_n\|f_n - P_n\|, \qquad n = 1, 2, \ldots. \tag{17}$$

For instance, we can define the numbers c_i inductively by means of the relation

$$c_i = \min\left\{\tfrac{1}{2}c_{i-1}\|f_{i-1} - P_{i-1}\|, \ldots, \frac{1}{2^{i-1}} c_1\|f_1 - P_1\|, i^{-2} M_i\right\}, \qquad i = 2, 3, \ldots.$$

For each n, let

$$F_n = \sum_{i=1}^{n} c_i f_i, \qquad \Pi_n = \sum_{i=1}^{n-1} c_i f_i + c_n P_n.$$

Thus, Π_n is a polynomial of degree N_{n-1}. Therefore

$$E_{N_{n-1}}(F_n) \leqslant \|F_n - \Pi_n\| \leqslant c_n\|f_n - P_n\|, \qquad n = 2, 3, \ldots. \tag{18}$$

On the other hand, let Q be a polynomial of degree N_{n-1} with $Q^{(k)}(x) \geqslant 0$ on $[-1, +1]$. By (13), $\Pi_n^{(k)}(0) = c_n P_n^{(k)}(0)$ is negative, and hence by Markov's inequality,

$$|\Pi_n^{(k)}(0)| \leqslant |\Pi_n^{(k)}(0) - Q^{(k)}(0)| \leqslant N_{n-1}^{2k}\|\Pi_n - Q\| \leqslant N_{n-1}^{2k}(\|F_n - Q\| + \|F_n - \Pi_n\|).$$

By (4),

$$(2n+2)\, N_{n-1}^{2k}\, c_n\|f_n - P_n\| = b_n c_n\|f_n - P_n\| < |\Pi_n^{(k)}(0)|$$
$$\leqslant N_{n-1}^{2k}\|F_n - Q\| + N_{n-1}^{2k}\, c_n\|f_n - P_n\|.$$

Since Q was an arbitrary polynomial subject to the condition $Q^{(k)}(x) \geqslant 0$,

$$E_{N_{n-1}}^{*}(F_n) \geqslant (2n+1)\, c_n\|f_n - P_n\|, \qquad n = 2, 3, \ldots.$$

The function f is k times continuously differentiable, as seen from (16), and $f^{(k)}(x) \geqslant 0$. The degrees of approximation of f differ from those of F_n by at most

$$\sum_{i=n+1}^{\infty} c_i \|f_i\| \leqslant \sum_{i=n+1}^{\infty} c_i \leqslant c_n \|f_n - P_n\|.$$

Hence

$$E_{N_{n-1}}(f) \leqslant 2c_n \|f_n - P_n\|,$$
$$E^*_{N_{n-1}}(f) \geqslant 2nc_n \|f_n - P_n\|,$$

and the theorem follows.

References

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2. G. G. Lorentz and K. L. Zeller, Degree of approximation by monotone polynomials I. *J. Approx. Th.* **1** (1968), 501–504.